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SUDDEN TWISTING OF A FLAT ANNULAR CRACK

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Abstract—The problem of a torque applied suddenly to the surface of a flat annular crack in an infinite elastic body is considered. The singular solution is equivalent to that of the sudden appearance of a crack in a body under torsion. Laplace and Hankel transforms are used to reduce the problem to a pair of triple integral equations. The solution to the triple integral equations is expressed in terms of a singular integral equation of the first kind with kernel improved by means of a contour integration on the Riemann surface. The singular stress distributions near the crack tip are obtained in closed form and the influences of the inertia, the ratio of the inner radius to the outer one and their interactions upon the dynamic stress-intensity factors are shown graphically.

1. INTRODUCTION

The study of dynamic elasticity, which involves the transient response of a crack-like imperfections to impact loads, has attracted attention of scientists because of its increasing application to fracture mechanics. At the crack, waves are reflected and refracted causing the local stress to increase beyond its corresponding value under static loads of the same magnitude. This could initiate the unstable motion of the crack and eventually the fracture of the structure. The dynamic response of a crack under the action of impact loads has been treated by many authors and many papers on the subject have been reviewed by the recent book of Sih[1]. However, these solutions are mostly limited to 2-Di problems and few results on the 3-Di analysis have been reported except the axisymmetric problems. The axisymmetric elastodynamic problem involving a penny-shaped crack in an infinite medium under torsional load has been considered by Sih and Embley[2]. A frequently encountered crack shape in embedded cracks is the banana-shaped crack which has become practically famous with a noted catastrophic fracture[3]. The stress-intensity factor at the critical region, i.e. at the midpoints of the waist of the crack, can be estimated by the stress-intensity factor of a flat annular crack. The problem of this type is a three-part mixed boundary value problem and is reduced to a solution of triple integral equations.

In this investigation, the impact response of a flat annular crack in an infinite solid undergoing the action of twisting is considered. Laplace and Hankel transforms are applied and mixed boundary value problem is formulated in the Laplace transform domain. The solution to the mixed boundary value problem is expressed in terms of a singular integral equation of the first kind. The convergence of the kernel has been improved by means of a contour integration on the Riemann surface. The solution of the singular integral equation is given in the form of the product of the series of Chebyshev polynomials of the first kind and their weight functions, and is obtained by solving a system of linear algebraic equations for the determination of unknown coefficients. A numerical Laplace inversion technique developed in [4] is used to obtain the solution in physical plane. By making the inner radius of the crack tend to zero, we solve the transient problem for a penny-shaped crack. It should be noted that we can't obtain the case of the penny-shaped crack by simply taking the inner radius to be zero. This case will be discussed in the Appendix. The value of the stress-intensity factor so derived agrees with the one by Sih and Embley[2]. The results can be used for the sudden appearance of a crack in a stressed medium under torque and for a transient torsional wave impinging on the flat annular crack. Results are presented by the dynamic stress-intensity factor and are shown graphically to demonstrate the influence of the time and geometric parameters.

2. STATEMENT OF THE PROBLEM

Consider an infinite homogeneous isotropic elastic solids that contains a flat annular crack of inner and outer radii a, b lying on the plane $z = 0$ in a cylindrical polar coordinate system (r, θ, z), as shown in Fig. 1. Let the components of the displacement in the r, θ, z directions be given by u_r, u_θ and u_z. For the torsional shear problem, u_r and u_z vanish everywhere and u_θ is a function of r, z and time t only:

$$u_r = u_z = 0, \quad u_\theta = u_\theta(r, z, t). \tag{1}$$

The corresponding stress field consists of two shear stresses

$$\sigma_{r\theta} = \mu r(u_\theta/r)_{,r}$$

$$\sigma_{\theta z} = \mu u_{\theta,z} \tag{2}$$

while all other components vanish. In eqns (2), μ stands for the shear modulas of elasticity of the solid medium and a comma denotes partial differentiation with respect to the coordinates.

Substituting eqns (2) into the equation of motion of elasticity in the θ-direction renders

$$(u_{\theta,r} + u_\theta/r)_{,r} + u_{\theta,zz} = u_{\theta,tt}/c_2^2 \tag{3}$$

where the shear wave velocity c_2 is defined by $c_2^2 = \mu/\rho$ with ρ being the mass density of the material. Equation (3) is to be solved subjected to zero initial conditions and the following boundary and symmetry conditions:

$$\sigma_{z\theta}(r, 0, t) = -\tau_0(r/b)H(t);\ a < r < b \tag{4}$$

$$u_\theta(r, 0, t) = 0;\ 0 \leq r \leq a,\ b \leq r \tag{5}$$

where τ_0 is a constant with the dimension of stress and $H(t)$ is Heaviside unit step function. Away from the crack, the tangential displacement is required to vanish, i.e. $u_\theta \to 0$ as $(r^2 + z^2)^{1/2} \to \infty$. Because of the symmetry condition across the plane $z = 0$ it is possible to consider the problem for the upper half space, $z \geq 0$.

3. METHOD OF SOLUTION

Define a Laplace transform pair by the equations

$$f^*(p) = \int_0^\infty f(t)\exp(-pt)\,\mathrm{d}t \tag{6}$$

$$f(t) = \frac{1}{2\pi i}\int_{Br} f^*(p)\exp(pt)\,\mathrm{d}p \tag{7}$$

where the second integral is integrated over the Bromwich path. The application of eqn (6) to (3) yields

$$u^*_{\theta,rr} + u^*_{\theta,r}/r - u^*_\theta/r^2 + u^*_{\theta,zz} = (p/c_2)^2 u^*_\theta \tag{8}$$

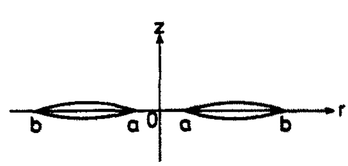

Fig. 1. An infinite elastic solid containing a flat annular crack.

in which $u_\theta^* = u_\theta^*(r, z, p)$. A Hankel transform is applied on eqn (8) and the result is

$$u_\theta^* = \int_0^\infty \alpha A(\alpha, p) \exp[-\gamma(\alpha) z] J_1(\alpha r)\, d\alpha \tag{9}$$

where J_1 is the first-order Bessel function of the first kind and

$$\gamma(\alpha) = \{\alpha^2 + (p/c_2)^2\}^{1/2}$$

In the Laplace transform plane, eqns (4) and (5) become

$$\sigma_{z\theta}^*(r, 0, p) = -\tau_0(r/bp);\ a < r < b \tag{10}$$

$$u_\theta^*(r, 0, p) = 0;\ 0 \leq r \leq a,\ b \leq r \tag{11}$$

Making use of eqns (2), (6), (7) and (9), eqns (10) and (11) render a pair of triple integral equations:

$$\int_0^\infty \alpha^2 A(\alpha, p) J_1(\alpha r)\, d\alpha = \int_0^\infty \alpha g(\alpha, p) A(\alpha, p) J_1(\alpha r)\, d\alpha + (\tau_0/\mu p)(r/b);\ a < r < b \tag{12}$$

$$\int_0^\infty \alpha A(\alpha, p) J_1(\alpha r)\, d\alpha = 0;\ 0 \leq r \leq a,\ b \leq r \tag{13}$$

where

$$g(\alpha, p) = \alpha - \gamma(\alpha). \tag{14}$$

If one defines

$$r(u_\theta^*/r)_{,r} = \varphi(r, p);\ z = 0,\ a < r < b\ = 0;\ z = 0,\ 0 \leq r \leq a,\ b \leq r \tag{15}$$

then with the help of the equation (9), $A(\alpha, p)$ is determined as

$$\alpha A(\alpha, p) = -\int_a^b t\varphi(t, p) J_2(\alpha t)\, dt \tag{16}$$

where J_2 is the second order Bessel function of the first kind. If we now substitute eqn (16) into eqn (12), after some manipulations, we have

$$\int_a^b t\varphi(t, p)\, [R_0(r, t) + R_1(r, t)]\, dt = -(\tau_0/\mu p)(r/b);\ a < r < b \tag{17}$$

where

$$\begin{aligned} R_0(r, t) &\equiv \int_0^\infty \alpha J_2(\alpha t) J_1(\alpha r)\, d\alpha \\ &= \frac{4}{\pi t^2}[K(t/r) - E(t/r)] + \frac{2}{\pi} \cdot \frac{1}{t^2 - r^2} E(t/r);\ t < r \\ &= \frac{4}{\pi r t}[K(r/t) - E(r/t)] + \frac{2}{\pi r t}\left[\frac{t^2}{t^2 - r^2} E(r/t) - K(r/t)\right];\ t > r \end{aligned} \tag{18}$$

$$R_1(r, t) = -\int_0^\infty g(\alpha, p) J_2(\alpha t) J_1(\alpha r)\, d\alpha. \tag{19}$$

Here, K and E are the complete elliptic integrals of the first and second kind, respectively. From eqn (13) and the definition (16) it is clear that the integral equation must be solved under the following single-valuedness condition:

$$\int_a^b \frac{1}{t}\,\varphi(t,p)\,\mathrm{d}t = 0. \tag{20}$$

The integral in (19) exists for all r, t in $[a, b]$ and can be converted into integral with fast rate of convergence. To evaluate the integral in (19), we consider the contour integrals:

$$I_{C1} = \oint_{C1} L(\gamma(k), k) H_2^{(1)}(kt)\, J_1(kr)\,\mathrm{d}k;\ r < t \tag{21}$$

$$I_{C2} = \oint_{C2} L(\gamma(k), k) H_2^{(2)}(kt)\, J_1(kr)\,\mathrm{d}k;\ r < t$$

in which

$$L(\gamma(k), k) \equiv k - \gamma(k) = k - \{k^2 + (p/c_2)^2\}^{1/2}. \tag{22}$$

In eqns (21), the contours $C1$, $C2$ are defined in Fig. 2 and $H_2^{(1)}$, $H_2^{(2)}$ are respectively, the second order Hankel functions of the first and second kind. The integrals in (21) satisfy Jordan's Lemma on the infinite quarter circles, so that,

$$I_{C2} = \int_0^\infty \{\alpha - \gamma(\alpha)\} H_2^{(1)}(\alpha t) J_1(\alpha r)\,\mathrm{d}\alpha + \int_\infty^{p/c_2} \{i\alpha - i\nu'(\alpha)\} H_2^{(1)}(i\alpha t) J_1(i\alpha r) i\,\mathrm{d}\alpha$$

$$+ \int_{p/c_2}^{0} \{i\alpha - \nu(\alpha)\} H_2^{(1)}(i\alpha t) J_1(i\alpha r) i\,\mathrm{d}\alpha + I_{1\epsilon} \tag{23}$$

$$I_{c2} = \int_0^\infty \{\alpha - \gamma(\alpha)\} H_2^{(2)}(\alpha t) J_1(\alpha r)\,\mathrm{d}\alpha + \int_\infty^{p/c_2} \{-i\alpha + i\nu'(\alpha)\} H_2^{(2)}(-i\alpha t) J_1(-i\alpha r)(-i)\,\mathrm{d}\alpha$$

$$+ \int_{p/c_2}^{0} \{-i\alpha - \nu(\alpha)\} H_2^{(2)}(-i\alpha t)\, J_1(-i\alpha r)(-i)\,\mathrm{d}\alpha + I_2\epsilon$$

where $\nu(\alpha) = (p^2/c_2^2 - \alpha^2)^{1/2}$ and $\nu'(\alpha) = (\alpha^2 - p^2/c_2^2)^{1/2}$, and $I_{1\epsilon}$, $I_{2\epsilon}$ are the residue contributions

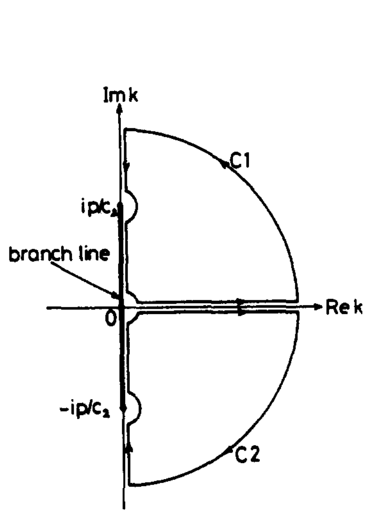

Fig. 2. Contours of integration for integrals in eqns (21).

from the upper and lower quarter-circles at the point $k = 0$ and are given by

$$I_{1\epsilon} \equiv \lim_{\epsilon\to 0}\int_{\pi/2}^{0} L(\gamma(\epsilon\, e^{i\theta}), \epsilon\, e^{i\theta})H_2^{(1)}(t\epsilon\, e^{i\theta})J_1(r\epsilon\, e^{i\theta})i\epsilon\, e^{i\theta}\, d\theta = (p/c_2)r/t^2$$

$$I_{2\epsilon} \equiv \lim_{\epsilon\to 0}\int_{-\pi/2}^{0} L(\gamma(\epsilon\, e^{i\theta}), \epsilon\, e^{i\theta})H_2^{(2)}(t\epsilon\, e^{i\theta})J_1(r\epsilon\, e^{i\theta})i\epsilon\, e^{i\theta} d\theta = (p/c_2)r/t^2. \tag{24}$$

Since $I_{C1} + I_{C2} = 0$, we get the relation,$I_1(\alpha r)$

$$\int_0^\infty \{\alpha - \gamma(\alpha)\}J_2(\alpha t)J_1(\alpha r)\, d\alpha$$
$$= \frac{2}{\pi}\left[\int_0^{p/c_2} \alpha K_2(\alpha t)I_1(\alpha r)\, d\alpha + \int_{p/c_2}^\infty \{\alpha - \nu'(\alpha)\}K_2(\alpha t)I_1(\alpha r)\, d\alpha\right] - (p/c_2)r/t^2;\ r < t. \tag{25}$$

The case for $r > t$ can be also found. Therefore $R_1(r, t)$ can be finally written as

$$R_1(r,t) = -\frac{2}{\pi}\left[\!\left[\int_0^{p/c_2} \alpha K_2(\alpha t)I_1(\alpha r)\, d\alpha + \int_{p/c_2}^\infty [\alpha - \{\alpha^2 - (p/c_2)^2\}^{1/2}]K_2(\alpha t)I_1(\alpha r)\, d\alpha\right]\!\right]$$
$$+ (p/c_2)r/t^2;\ r < t$$
$$= \frac{2}{\pi}\left[\!\left[\int_0^{p/c_2} \alpha I_2(\alpha t)K_1(\alpha r)\, d\alpha + \int_{p/c_2}^\infty [\alpha - \{\alpha^2 - (p/c_2)^2\}^{1/2}]I_2(\alpha t)K_1(\alpha r) d\alpha\right]\!\right];\ t < r \tag{26}$$

where I_1, I_2, K_1 and K_2 represent the usual modified Bessel functions. The kernels $R_1(r, t)$ given in the form of eqns (26) will allow their numerical evaluation with good accuracy.

The kernel $R_0(r, t)$ has Cauchy-type singularities and logarithmic singularities which are assumed to be separated. We thus have

$$R_0(r,t) = \frac{1}{t^2\pi}\left[\frac{r}{t-r} - \frac{3}{2}\log\left|\frac{2(t-r)}{b-a}\right| + M_0(r,t)\right] \tag{27}$$

where

$$M_0(r,t) = \left\{\frac{r}{t+r} - 2(t/r)\right\}E(r/t) + \frac{r}{t-r}\{E(r/t) - 1\} + 2(t/r)K(r/t) + \frac{3}{2}\log\left|\frac{2(t-r)}{b-a}\right|;\ t < r$$

$$= \left\{\frac{r(r/t)}{t+r} - 2\right\}E(t/r) + \frac{r}{t-r}\{(r/t)E(t/r) - 1\} + 4K(t/r) + \frac{3}{2}\log\left|\frac{2(t-r)}{b-a}\right|;\ t > r. \tag{28}$$

For the sake of convenience, we perform the following non-dimensionalization:

$$R = r/b = \frac{1}{2}(1 - a_0)s + \frac{1}{2}(1 + a_0)$$

$$T = t/b = \frac{1}{2}(1 - a_0)\tau + \frac{1}{2}(1 + a_0)$$

$$a_0 = a/b \tag{29}$$

$$P = bp/c_2$$

$$\Phi(\tau, P) = \varphi(t,p)/(\tau_0/\mu p)T.$$

With the help of eqns (28) and (29), the revised singular integral equation of the first kind (17) and single-valuedness condition (20) are shown as

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{\tau-s}-\frac{3(1-a_0)}{4R}\log|\tau-s|+K(s,\tau)\right]\Phi(\tau,P)\,\mathrm{d}\tau=-1 \tag{30}$$

$$\int_{-1}^{1}\Phi(\tau,P)\,\mathrm{d}\tau=0 \tag{31}$$

in which the Fredholm kernel $K(s,\tau)$ is bounded in closed interval $-1<s,\tau<1$ and is given by

$$k(s,\tau)=\frac{1-a_0}{2R}[M_0(r,t)+T^2M_1(s,\tau)] \tag{32}$$

$$M_1(s,\tau)=-2P^2\left[\int_0^1\alpha K_2(\alpha PT)I_1(\alpha PR)\mathrm{d}\alpha+\int_1^\infty\{\alpha-(\alpha^2-1)^{1/2}\}K_2(\alpha PT)I_1(\alpha PR)\mathrm{d}\alpha\right]+\pi PR/T^2;\ s<\tau$$

$$=2P^2\left[\int_0^1\alpha I_2(\alpha PT)K_1(\alpha PR)\,\mathrm{d}\alpha+\int_1^\infty\{\alpha-(\alpha^2-1)^{1/2}\}I_2(\alpha PT)K_1(\alpha PR)\,\mathrm{d}\alpha\right];\tau<s. \tag{33}$$

Thus the problem is reduced to the solution of the singular integral equation (30) under the additional condition (31).

The solution of the set of integral equations (30) and (31) as given in [5], is

$$\Phi(\tau,P)=\frac{1}{(1-\tau^2)^{1/2}}\sum_{n=1}^{\infty}A_nT_n(\tau) \tag{34}$$

where $T_n(\tau)$ are Chebyshev polynomials of the first kind and A_n are unknown constants obtained from the following infinite system of linear algebraic equations:

$$\sum_{n=1}^{\infty}(\delta_{kn}+\alpha_{kn}+\beta_{kn})A_n=-\delta_{1k}. \tag{35}$$

In eqns (35), δ_{kn} is Kronecker delta and

$$\alpha_{kn}=\frac{3(1-a_0)}{2n\pi}\int_{-1}^{1}\frac{1}{R}\,T_n(s)\,U_{k-1}(s)(1-s^2)^{1/2}\,\mathrm{d}s \tag{36}$$

$$\beta_{kn}=\frac{2}{\pi^2}\int_{-1}^{1}U_{k-1}(s)(1-s^2)^{1/2}\,\mathrm{d}s\int_{-1}^{1}T_n(\tau)K(s,\tau)\frac{1}{(1-\tau^2)^{1/2}}\,\mathrm{d}\tau \tag{37}$$

where $U_{k-1}(s)$ are Chebyshev polynomials of the second kind. All the integrals in eqns (36) and (37) are Gauss–Chebyshev type and may easily be evaluated by using the proper quadrature formulas [6].

4. TRANSIENT STRESS DISTRIBUTION AROUND THE CRACK

From the fracture mechanics point of view, the desired information is the singular stresses near the crack tip. Instead of inverting the complete stress field back to the physical plane, only the portion of the asymptotic stress field near the crack periphery needs to be inverted. The integral expression for the Laplace transform of the stresses can be obtained by substituting eqn (9) into eqns (2). Noting that the integrands are finite and continuous for any given values of α, the divergence of the integral near the crack tip must be due to the behaviour of the integrand as the integration variable $\alpha\to\infty$. Making use of eqns (16), (29) and (34), the singular portion of the stress field in the transformed plane may be obtained by expanding the integral expressions asymptotically for large value of α and then carrying out the integration. Applying the theorem [7] on the behaviour of Cauchy integral near the ends of the path of integration and then the Laplace inversion theorem, the local stress field near the crack is obtained as:

$$\sigma_{r\theta}(r, z, T) \sim \frac{K_{3a}(T)}{(2\rho_a)^{1/2}} \cos(\theta_a/2) - \frac{K_{3b}(T)}{(2\rho_b)^{1/2}} \sin(\theta_b/2)$$

$$\sigma_{z\theta}(r, z, T) \sim \frac{K_{3a}(T)}{(2\rho_a)^{1/2}} \sin(\theta_a/2) + \frac{K_{3b}(T)}{(2\rho_b)^{1/2}} \cos(\theta_b/2) \tag{38}$$

where K_{3a} and K_{3b} stand for the dynamic stress-intensity factors at the inner and the outer tips of the crack, respectively, and are defined by

$$K_{3a}(T) = \lim_{r\to a^-} \{2(a-r)\}^{1/2} \sigma_{z\theta}(r, 0, T) = \frac{\tau_0 b^{1/2}}{2^{1/2}} a_0(1-a_0)^{1/2} \frac{1}{2\pi i} \int_{Br} \frac{1}{P} \sum_{n=1}^{\infty} (-1)^n A_n e^{PT} dP$$

$$K_{3b}(T) = \lim_{r\to b^+} \{2(r-b)\}^{1/2} \sigma_{z\theta}(r, 0, T) = -\frac{\tau_0 b^{1/2}}{2^{1/2}} (1-a_0)^{1/2} \frac{1}{2\pi i} \int_{B_r} \frac{1}{P} \sum_{n=1}^{\infty} A_n e^{PT} dP. \tag{39}$$

In eqns (38), $T = c_2 t/b$ is the normalized time, and ρ_a, θ_a and ρ_b, θ_b are the polar coordinates defined as

$$\rho_a = \{(r-a)^2 + z^2\}^{1/2}, \quad \theta_a = \tan^{-1}\left\{\frac{z}{r-a}\right\}$$

$$\rho_b = \{(r-b)^2 + z^2\}^{1/2}, \quad \theta_b = \tan^{-1}\left\{\frac{z}{r-b}\right\}. \tag{40}$$

The equivalence of inverting only the near field solution to that of the entire stress field for obtaining the dynamic stress-intensity factor has been established in [8] by using a modified Cagniard–De Hoop method. Note that the same angular distribution and inverse square root singularity are recovered for the dynamic problem.

The infinite system of simultaneous equations (35) may be solved numerically to determine the coefficients $A_n (n = 1, 2, \ldots)$. Once this is done, the integrals in eqns (39) must be determined in order to evaluate the stress-intensity factor. In this paper, numerical method is employed for the Laplace inversion. The formula [4] used is as follows:

$$F^*(P/\delta) \equiv \int_0^{\infty} F(T) e^{-(P/\delta)T} dT \doteqdot \delta \sum_{j=1}^{N} w_j x_j^{P-1} F(-\delta \log x_j) \tag{41}$$

where δ is a parameter to determine the time scale, the $x_j (j = 1, \ldots, N)$ are the zeros of the shifted Legendre polynomials $P_N\ (1-2x)$ and the w_j are a set of weights which are given by

$$w_j = -\frac{1}{2} \int_0^1 \frac{P_N(1-2x)}{(x-x_j)[P'_N(1-2x)]_{x=x_j}} dx. \tag{42}$$

In eqn (42), $P_N\ (x)$ are the Legendre polynomials of order N and $P'_N\ (1-2x)$ is used for $dP_N\ (1-2x)/d(1-2x)$.

5. NUMERICAL RESULTS AND DISCUSSIONS

Numerical results have been calculated for the dynamic stress-intensity factor as a function of time for various ratios of a_0. Note that letting $P \to 0$ in eqns (35) yields the corresponding solution for the static case. As $T \to \infty$ and $a_0 \to 0$, the dynamic stress-intensity factor K_{3b} at the outer tip of the crack tends to the static solution $K_{3s} = (4/3\pi)\, \tau_0 b^{1/2}$ for the penny-shaped crack of radius b. The dynamic stress-intensity factors are normalized by the static solution K_{3s}. In the calculated results, it is found that the value of N needed to achieve a particular level of accuracy is strongly dependent a_0 and the truncation after $N = 16$, 12, 10, 8 and 6 gives practically adequate results at any desired transform parameter P for $a_0 = 0.1$, 0.3, 0.5, 0.7 and 0.9, respectively.

The normalized dynamic stress-intensity factor K_{3a}/K_{3s} at the inner tip of the crack is plotted in Fig. 3 as a function of the time variable T for various values of a_0 ratio. The same kind of results for K_{3b}/K_{3s} at the outer tip of the crack is shown in Fig. 4. Results in Figs. 3 and 4 are obtained for seven integral values of P from 1 to 7, the parameter $N = 7$ of the numerical Laplace inversion and the parameters $\delta = 0.5$, 0.6, 0.7, 0.8, 0.9 and 1.0 of the time scale. The dynamic stress-intensity factor increases quickly with time, reaching a peak and then decrease in magnitude oscillating around its corresponding static value. The same trend has been also observed for the case of the penny-shaped crack[2]. The dynamic stress-intensity factor K_{3b}/K_{3s} for $a_0 = 0.1$ is almost coincident with the result of the penny-shaped crack in an infinite body. The solution obtained in the preceding sections tends to that of the penny-shaped crack as $a_0 \to 0$, but we can't obtain the case of the penny-shaped crack by simply taking a_0 to be zero. But we can obtain this case by extending the definitions of the unknown function and the kernels[9], which is given in the Appendix. The numerical result for $a_0 = 0$ given in the Appendix coincides the result by Sih and Embley[2]. The peak value of K_{3a}/K_{3s} occurs later in time as a_0 is decreased. It is observed that the combined impact and geometry effects can increase the peak values of the stress-intensity factor K_{3a}/K_{3s} by approximate 2 and 5% for $a_0 = 0.3$, 0.5 and can decrease them by approximate 31, 7 and 42% for $a_0 = 0.1$, 0.7, 0.9, respectively. And also the peak value of K_{3b}/K_{3s} appears to be higher and occurs later in time as a_0 is decreased. For $a_0 = 0.1$, 0.3 and 0.5, the combined impact and geometry effects can increase the peak values of the stress-intensity factor K_{3b}/K_{3s} by approximate 20, 19 and 12%. For $a_0 = 0.7$ and 0.9, the opposite effects are observed and the effects can decrease them by approximate 3 and 41%. However, by

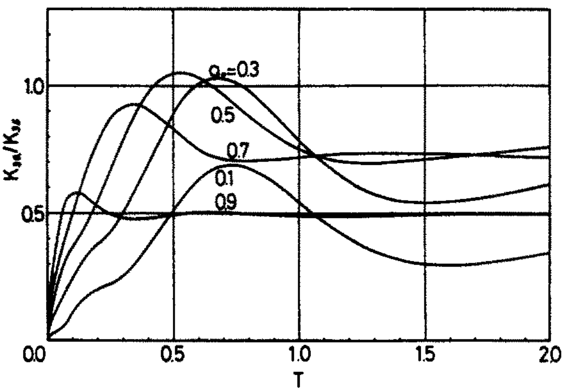

Fig. 3. Variation of dynamic stress-intensity factor at the inner tip of the crack with time.

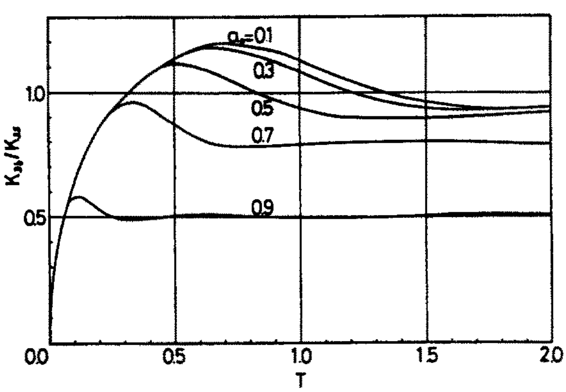

Fig. 4. Variation of dynamic stress-intensity factor at the outer tip of the crack with time.

comparing the corresponding static stress-intensity factors $K_{3a}/K_{3s} = 0.400, 0.658, 0.759, 0.727$ and 0.493 and $K_{3b}/K_{3s} = 0.999, 0.985, 0.931, 0.800$ and 0.506 for $a_0 = 0.1, 0.3, 0.5, 0.7$ and 0.9 with Figs. 3 and 4, it is observed that the inertia effects can increase the peak values of the stress-intensity factor at the inner tip by approximate 73, 55, 39, 29 and 17% and the peak values of the stress-intensity factor at the outer tip by approximate 20, 20, 20, 21 and 17%, respectively. For small a_0, the intertia effects for the stress-intensity factor at the inner tip become large. The inertia effects for the stress-intensity factor at the outer tip don't almost depend on a_0. In any case, the inertia effects increase the peak values of the stress-intensity factor and must be accounted for. The maximum values of K_{3a}/K_{3s} for $a_0 = 0.1, 0.3, 0.5\ 0.7$ and 0.9 are produced almost at the same time with those of K_{3b}/K_{3s}, respectively. Since the values of K_{3b}/K_{3s} for $a_0 = 0.1, 0.3, 0.5$, and 0.9 are always larger than those of K_{3a}/K_{3s}, the annular crack transforms into an external crack. After that the material will be broken into two pieces.

In conclusion, the impact response of a flat annular crack in an infinite solid is obtained. The results are expressed in terms of the dynamic stress-intensity factor. Depending on the radius ratio a_0, the peak values of the dynamic stress-intensity factor are always higher than the static values and are produced at time much sooner than that as found in[2] for the case of the penny-shaped crack.

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APPENDIX

Transient stress-intensity factor for a penny-shaped crack

We will consider the solution for a penny-shaped crack which was solved by Sih and Embley[2] by transforming the dual integral equations into the Fredholm integral equation of the second kind. The penny-shaped crack solution can be obtained from the annular crack solution by extending the definitions of the unknown function and the kernels into $-b < r$, $t < 0$ in an appropriate way[9]. The nature of axial symmetry of the problem suggests that $\varphi(t, p) = -\varphi(-t, p)$[9]. Referring to [9], if we extend the definitions of the function $\varphi(t, p)$ and the kernels from $0 < r, t < b$ into $-b < r, t < b$, and if we define the dimensionless variables

$$s = r/b, \quad \tau = t/b \tag{43}$$

(17) and (20) may be expressed as

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{\tau - s} - \frac{3}{2s}\log|\tau - s| + K_0(s, \tau)\right]\Phi_0(\tau, P)\mathrm{d}\tau = -1 \tag{44}$$

$$\int_{-1}^{1}\Phi_0(\tau, P)\,\mathrm{d}\tau = 0 \tag{45}$$

where

$$\Phi_0(\tau, P) = \varphi(t, p)/(\tau_0/\mu p)\tau \tag{46}$$

$$K_0(s, \tau) = L_0(s, \tau) + \frac{\pi\tau|\tau|}{2s}L_1(s, \tau) \tag{47}$$

$$L_0(s,\tau)=-\frac{\tau+s}{|\tau s|}E(|\tau|/|s|)+\frac{(|s|/|\tau|)E(|\tau|/|s|)-1}{\tau-s}$$

$$+\frac{2(s+\tau)K(|\tau|/|s|)}{|\tau s|}+\frac{3}{2s}\log|\tau-s|;\quad |\tau|<|s|$$

$$=-\frac{\tau+s}{s^2}E(|s|/|\tau|)+\frac{E(|s|/|\tau|)-1}{\tau-s}+\frac{(\tau+s)K(|s|/|\tau|)}{s^2} \tag{48}$$

$$+\frac{3}{2s}\log|\tau-s|;|\tau|>|s|$$

$$L_1(s,\tau)=\frac{2}{\pi}P^2\left[\int_0^1 \alpha I_2(\alpha P|\tau|)K_1(\alpha P|s|)\,d\alpha\right.$$

$$\left.+\int_1^\infty \{\alpha-(\alpha^2-1)^{1/2}\}I_2(P\alpha|\tau|)K_1(P\alpha|s|)d\alpha\right];|\tau|<|s| \tag{49}$$

$$=-\frac{2}{\pi}P^2\left[\int_0^1 \alpha K_2(\alpha P|\tau|)I_1(\alpha P|s|)\,d\alpha\right.$$

$$\left.+\int_1^\infty \{\alpha-(\alpha^2-1)^{1/2}\}K_2(P\alpha|\tau|)I_1(P\alpha|s|)\,d\alpha\right]+P|s|/|\tau|^2;|\tau|>|s|$$

To solve (44), we let

$$\Phi_0(\tau,P)=\frac{1}{(1-\tau^2)^{1/2}}\sum_{n=1}^{\infty}B_nT_{2n-1}(\tau) \tag{50}$$

where B_n are unknown constants. Since $\Phi_0(\tau,P)$ is an odd function, the condition (45) is satisfied. From (44) and (50), we obtain

$$\sum_{n=1}^{\infty}(\delta_{kn}+\alpha_{kn0}+\beta_{kn0})B_n=-\delta_{1k} \tag{51}$$

where

$$\alpha_{kn0}=\frac{3}{(2n-1)\pi}\int_{-1}^{1}\frac{1}{s}T_{2n-1}(s)\,U_{2k-2}(s)(1-s^2)^{1/2}\,ds \tag{52}$$

$$\beta_{kn0}=\frac{2}{\pi^2}\int_{-1}^{1}U_{2k-2}(s)(1-s^2)^{1/2}\,ds\int_{-1}^{1}T_{2n-1}(\tau)K_0(s,\tau)\frac{1}{(1-\tau^2)^{1/2}}\,d\tau. \tag{53}$$

The stress-intensity factor is found to be

$$K_{3b}(T)=\lim_{r\to b^+}\{2(r-b)\}^{1/2}\,\sigma_{z\theta}(r,0,T) \tag{54}$$

$$=-\tau_0 b^{1/2}\frac{1}{2\pi i}\int_{Br}\frac{1}{P}\sum_{n=1}^{\infty}B_n\,e^{PT}\,dP$$